Electrostatic energy in dielectric Media

Let, the charge q is on the positive plate, so that the potential difference is $V = q/C$. The total work, then to go from $q = 0$ to $q = Q$ is,

$$W = \int_0^Q \frac{q}{C}\, dq = \frac{1}{2} C V^2 \quad \text{(electrostatic energy in Vaccum)}$$

where, $Q = CV$.

If the capacitor is filled with linear dielectric, its capacitance exceeds the vaccum value by a factor of the dielectric Constant.

$$(C = \epsilon_r C_{vac.})$$

Hence, work will be increased by same factor. The Reason is that you have to pump on more free charge to achieve a given potential, bcoz part of the field is cancelled by the bound charges.

(Hence, for a dielectric filled capacitor,

$$W = \frac{\epsilon_0}{2} \int \epsilon_r \cdot E^2 \, d\tau = \frac{1}{2} \epsilon_0 \epsilon_r \int |E|^2 \, d\tau \quad \text{——— (1)}$$

~~bcoz~~ Since, energy stored in electrostatic system is

$$W = \frac{\epsilon_0}{2} \int |E|^2 \, d\tau \quad \text{——— (2)}$$

then eq^n (1) reduces to;

$$W = \frac{1}{2} \int \underline{D} \cdot \underline{E} \, d\tau$$

(in presence of linear dielectrics)

To prove it, suppose the dielectric material is fixed in position and we bring in the free charge a, bit at a time. As ρ_f is increased by an amount $\Delta\rho_f$, the polarization will change; the work done on the Incremental free charge:

$$\Delta W = \int (\Delta\rho_f) V \, d\tau$$

Since, $\nabla\cdot\underline{D} = \rho_f$, $\Delta\rho_f = \nabla\cdot(\Delta\underline{D})$ where $\Delta\underline{D}$ is the resulting change in $\underline{D}$, so

$$\Delta W = \int [\nabla\cdot(\Delta\underline{D})] V \, d\tau$$

Now,

$$\nabla\cdot[(\Delta\underline{D})V] = [\nabla\cdot(\Delta\underline{D})]V + \Delta\underline{D}\cdot(\nabla V)$$

and Hence,

$$\Delta W = \int_V \nabla\cdot[(\Delta\underline{D})V] d\tau - \int_V (\Delta\underline{D})\cdot(\nabla V) d\tau$$

$\because \underline{E} = -\nabla V$

$$\Delta W = \int_V \nabla\cdot[(\Delta\underline{D})V] \, d\tau + \int (\Delta\underline{D})\cdot\underline{E} \, d\tau$$

using div. Theorem

$$\int_V \nabla\cdot\underline{A} \, d\tau = \oint_S \vec{A}\cdot\hat{n} \, ds$$

Then,

$$\Delta W = \int_S (\Delta\underline{D}\cdot V)\cdot\hat{n} \, ds + \int (\Delta\underline{D})\cdot\underline{E} \, d\tau$$

then, surface integral will be vanish

$$\Delta W = \int (\Delta\underline{D})\cdot\underline{E} \, d\tau \quad \text{——— (3)}$$

Now, If the medium is a linear dielectric, then $\vec{D} = \epsilon\vec{E}$, So

$$\frac{1}{2}\Delta(\underline{D}\cdot\underline{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta\underline{E})\cdot\underline{E} \qquad \cancel{2}\cdot\epsilon\cdot E \, dE$$

$$\text{or, } \frac{1}{2}\Delta(\underline{D}\cdot\underline{E}) = (\Delta\underline{D})\cdot\underline{E}$$

Thus, by (3)

$$\Delta W = \Delta\left(\frac{1}{2}\int \underline{D}\cdot\underline{E} \, d\tau\right)$$

Then, the total work done,

$$\boxed{W = \frac{1}{2}\int \underline{D}\cdot\underline{E} \, d\tau} \quad \text{——— (4)}$$

eqⁿ (2) is apply in case of vacuum and eqⁿ (4) is apply in presense of dielectric. eqⁿ (2) does not include the work done in streching and string the dielectric moleculer it does not inclue the spring energy $\frac{1}{2}kx^2$. With the unpolarized dielectric in place, we bring in the free charges one by one allowing the dielectric to respond as it sees fit. Since free charge is actually push around then we get eqⁿ (4).

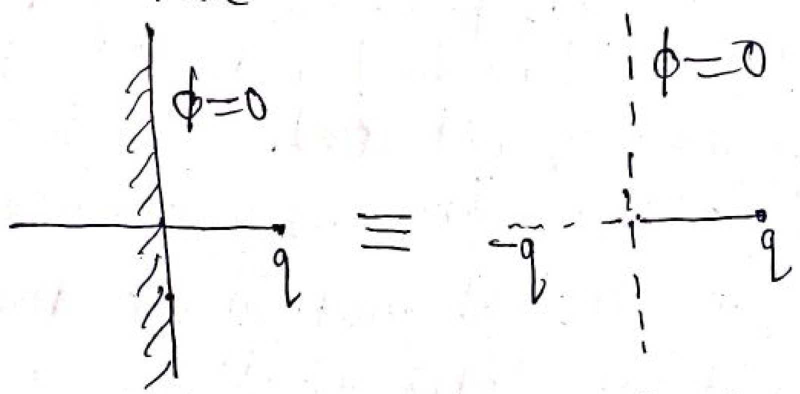
The total energy of the system

$$\boxed{W_{total} = W_{free} + W_{bound} + W_{spring}}$$

In Method(2) last two are equal and opposite, thus Method (2) in Calculating W_{free}, actually delivers W_{tot}, whereas Method(1), by Calculating $W_{free} + W_{bound}$, leaves out spring.

Method of Images:- The method of images concerns itself with one or more point charges in the presence of boundary surfaces. ex:- Conductors either grounded or held at fixed potential. Under condⁿ it is possible that suitable charges of appropriate magnitude external to region of interests can produced boundary condⁿ. These charges are called image charge. and replacement of actual prob with boundaries by an enlarged region with image charges is called Method of images. Image charges must be external to the volume of interests So ~~therefor~~ their potential must be Sol^n inside the volume particular integral is Sol^n of Poisson eqs: is provided by (Poiss sum of potential of charges inside the volume.

Ex:- A point charge is located in front of this ∞ plane Conductor at zero potential



(actual prob.) fig.(a)

(own prob.) equivalent Image Prob. fig.(b)

It is clear that this is equivalent ~~the~~ original prob. of actual charge and ~~equivalen~~ equal and opposite Charge ~~of~~ located at mirror image point ~~define in the position of Conductor~~ behind the plane defined by the position of the Conductor.